'Extended' Bradley-Terry models

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Pair-comparison studies

Sport: player i beats player j

Psychometrics: object i is preferred to object j

Sport (etc.): interest in *players* and their attributes

Psychometrics (etc.): interest in *judges* (*subjects*) and their attributes

Bradley-Terry model

The basic model:

$$\mathrm{pr}(i \text{ beats } j) = \frac{\alpha_i}{\alpha_i + \alpha_j},$$

with α_i the relative 'ability' of object i.

Work with *log* abilities:

$$\begin{aligned} \mathrm{logit}[\mathrm{pr}(i \text{ beats } j)] &= \log(\alpha_i) - \log(\alpha_j) \\ &= \lambda_i - \lambda_j. \end{aligned}$$

Extensions?

We will focus here on three possible directions from the basic model:

1. (Log-)abilities λ_i determined/predicted by object covariate vector x_i.
2. $\lambda_i \to \lambda_{ik}$: the ability of object i *varies* between different comparisons k.
3. i versus j, no preference? ('tied' comparisons)

'Structured' Bradley-Terry model

$$\begin{aligned} \lambda_i &= f_i(\beta) + U_i \\ &= \sum_r \beta_r x_{ir} + U_i \qquad \text{(for example)} \end{aligned}$$

- attributes of objects/players predict ability
- U_i is random error, with variance σ^2, say — needed in order to allow for imperfect prediction
- $\Rightarrow$ complex random effects model, with linear predictor

$$\sum_r (x_{ir} - x_{jr})\beta_r + (U_i - U_j)$$

Ability varying between comparisons

$$\lambda_i \to \lambda_{ik}$$

e.g., time-varying covariates,

$$\lambda_{ik} = \sum_r \beta_r x_{ikr} + U_i$$

e.g., subject-specific abilities,

$$\lambda_{ik} = \lambda_{is},$$

where $s = s(k)$ identifies the subject who makes comparison k.

e.g., abilities predicted by subject covariates,

$$\lambda_{is} = \sum_t \gamma_{it} z_{st} + E_{is}$$

Ability varying between comparisons (continued)

e.g., still with abilities λ_{is} varying between subjects, a particular form likely to be useful is *multiplicative* interaction,

$$\lambda_{is} = \lambda_i \exp\left(\sum_t \gamma_t z_{st}\right) + E_{is}$$

This last form is not yet implemented in the *BradleyTerry2* package; it will require features from the *gnm* (generalized nonlinear models) package.

Ties

What to do when neither i nor j is preferred?

Elaborate the Bradley-Terry model? (Rao and Kupper, 1967; Davidson, 1970)

A crude alternative approach/approximation:

$$\text{tie} = \text{half a `win' for each of } i \text{ and } j$$

Suggests a generalization: half → some other fraction?

Implementation in *R*: The *BradleyTerry2* package

Main new features

- flexible formula interface to modelling fitting function `BTm()`: allows object-specific, subject-specific, contest-specific variables and random effects [limited implementation]
- efficient data management of multiple data frames

Best of original *BradleyTerry* package

- translation of formula to appropriate design matrix
- methods for fitted model object, e.g. anova, BTabilities
- missing data handling

CEMS Data

The `CEMS` data (Dittrich et al, 1998) concern the preferences of students in selecting a school from the Community of European Management Schools for their international visit.

- 6 CEMS schools are covered in the survey
- students were to choose between each pair of schools (ties allowed)
- further data collected on students e.g. type of degree, language skills

Data Structure

```
> library(BradleyTerry2); data(CEMS); str(CEMS)
List of 3
 $ preferences:'data.frame': 4545 obs. of  8 variables:
  ..$ student : num [1:4545] 1 1 1 1 1 1 1 1 1 1 ...
  ..$ school1 : Factor w/ 6 levels "Barcelona","London",..: 2 2 4
  ..$ school2 : Factor w/ 6 levels "Barcelona","London",..: 4 3 3
  ..$ win1    : num [1:4545] 1 1 NA 0 0 0 1 1 0 1 ...
  ...
 $ students   :'data.frame': 303 obs. of  8 variables:
  ..$ STUD: Factor w/ 2 levels "other","commerce": 1 2 1 2 1 1 1 2
  ..$ ENG : Factor w/ 2 levels "good","poor": 1 1 1 1 2 1 1 1 2 1
  ...
 $ schools    :'data.frame': 6 obs. of  7 variables:
  ..$ Barcelona: num [1:6] 1 0 0 0 0 0
  ..$ London   : num [1:6] 0 1 0 0 0 0
...
```


Model Specification

Model specifiation is controlled by four arguments to `BTm()`

- `outcome` a binomial response as accepted by `glm()`.
- `player1, player2` specify the players in each contest and any other player-specific contest variables in data frames with the same attributes.
- `id` the name of the factor in `player1`/`player2` that gives the identity of the player.
- `formula` a one-sided formula for player ability.

Standard Bradley Terry Model

A Bradley-Terry model with a separate ability for each player can be specified as follows

```
> standardBT <-  BTm(outcome = cbind(win1.adj, win2.adj),
      player1 = data.frame(school = school1),
      player2 = data.frame(school = school2),
      id = "school", formula = ~ school,
      refcat = "Stockholm",
      data = CEMS$preferences)
```

Or we can use the default id, ".."

```
> standardBT <-  BTm(outcome = cbind(win1.adj, win2.adj),
      player1 = school1, player2 = school2,
      formula = ~ .., refcat = "Stockholm",
      data = CEMS$preferences)
```


Model Summaries

For models with no random effects, BTm returns an object which is essentially a "glm" object, hence the usual model summaries can be obtained, e.g. print():

```
Bradley Terry model fit by glm.fit

Call:  BTm(outcome = cbind(win1.adj, win2.adj), player1 = school1,
player2 = school2, formula = ~.., refcat = "Stockholm",
data = CEMS$preferences)

Coefficients:
..Barcelona      ..London      ..Milano       ..Paris  ..St.Gallen
     0.5379        1.5975        0.3878        0.9064       0.5251

Degrees of Freedom: 4454 Total (i.e. Null);  4449 Residual
  (91 observations deleted due to missingness)
Null Deviance:     5499
Residual Deviance: 4929   AIC: 5854
Warning message:
In eval(expr, envir, enclos) : non-integer counts in a binomial glm!
```


Object and Subject Variables

The final model in Dittrich et al, incorporating interactions with subject-covariates, can be estimated as follows

```
> interactionBT <-  BTm(outcome = cbind(win1.adj, win2.adj),
      player1 = school1, player2 = school2,
      formula = ~ .. +
      WOR[student] * LAT[..] +
      DEG[student] * St.Gallen[..] +
      STUD[student] * (Paris[..] + St.Gallen[..]) +
      ENG[student] * St.Gallen[..] +
      FRA[student] * (London[..] + Paris[..]) +
      SPA[student] * Barcelona[..] +
      ITA[student] * (London[..] + Milano[..]) +
      SEX[student] * Milano[..],
      refcat = "Stockholm", data = CEMS)
```


Interaction Model

```
> summary(interactionBT)$coef[, 1:2]/1.75

                                   Estimate Std. Error
..Barcelona                       1.0363917 0.10184195
..London                          1.2734839 0.10523535
..Milano                          1.1136211 0.10030192
..Paris                           0.6453467 0.05797807
..St.Gallen                       0.2487781 0.05663021
WOR[student]yes:LAT[..]           0.5933091 0.12278686
DEG[student]yes:St.Gallen[..]     0.2726479 0.06875424
STUD[student]commerce:Paris[..]   0.4073965 0.07352900
St.Gallen[..]:STUD[student]commerce -0.1984449 0.07089058
St.Gallen[..]:ENG[student]poor    0.1449582 0.07241576
FRA[student]poor:London[..]      -0.1607138 0.07519284
Paris[..]:FRA[student]poor       -0.7142351 0.07132559
SPA[student]poor:Barcelona[..]   -0.8409595 0.10336192
London[..]:ITA[student]poor      -0.2967857 0.10342156
ITA[student]poor:Milano[..]      -0.9603892 0.10386091
Milano[..]:SEX[student]male      -0.1743107 0.06848606
```


Baseball Data

The baseball data (Agresti, 2002) gives the results for 7 teams of the Eastern Division of the American League during the 1987 season:

```
> str(baseball)

'data.frame': 42 obs. of  4 variables:
 $ home.team: Factor w/ 7 levels "Baltimore","Boston",..: 5 5 5 5 5
 $ away.team: Factor w/ 7 levels "Baltimore","Boston",..: 4 7 6 2 3
 $ home.wins: int  4 4 4 6 4 6 3 4 4 6 ...
 $ away.wins: int  3 2 3 1 2 0 3 2 3 0 ...
```


Standard Bradley-Terry Model

```
> (baseballModel1 <- BTm(cbind(home.wins, away.wins), home.team,
away.team, data = baseball, id = "team"))

Bradley Terry model fit by glm.fit

Call:  BTm(outcome = cbind(home.wins, away.wins),
 player1 = home.team, player2 = away.team, id = "team",
 data = baseball)

Coefficients:
   teamBoston  teamCleveland    teamDetroit  teamMilwaukee
       1.1077         0.6839         1.4364         1.5814
 teamNew York     teamToronto
       1.2476         1.2945

Degrees of Freedom: 42 Total (i.e. Null);  36 Residual
Null Deviance:     78.02
Residual Deviance: 44.05   AIC: 140.5
```

Player-specific Contest Variables

```
> baseball$home.team <- data.frame(team = baseball$home.team,
+                                  at.home = 1)
> baseball$away.team <- data.frame(team = baseball$away.team,
+                                  at.home = 0)
> baseballModel2 <- update(baseballModel1,
+                          formula = ~ team + at.home)

...
Coefficients:
   teamBoston  teamCleveland    teamDetroit  teamMilwaukee
       1.1438         0.7047         1.4754         1.6196
 teamNew York    teamToronto        at.home
       1.2813         1.3271         0.3023

Degrees of Freedom: 42 Total (i.e. Null);  35 Residual
Null Deviance:      78.02
Residual Deviance: 38.64   AIC: 137.1
```

Extended Bradley-Terry models
└Implementation in R: The BradleyTerry2 package

Comparing Models

```
> anova(baseballModel1, baseballModel2)

Analysis of Deviance Table

Response: cbind(home.wins, away.wins)

Model 1: ~team
Model 2: ~team + at.home
  Resid. Df Resid. Dev Df Deviance
1        36     44.053
2        35     38.643  1   5.4106
```

Extended Bradley-Terry models
└Implementation in R: The BradleyTerry2 package

Springall Data

The `springall` data (Springall, 1973) gives the results of an experiment in which assessors were asked to determine which of two samples had the lesser flavour strength.

Samples were determined by a 3 x 3 factorial design, with factors flavour contentration and gel concentration.

The aim of the experiment was to describe the response surface over the two factors.

Extended Bradley-Terry models
└Implementation in R: The BradleyTerry2 package

Random Effects

The flavour strength over the design region can be modelled by a second order response surface model, with random effects to allow for variation between samples with the same covariates:

```
> springall.model <- BTm(cbind(win.adj, loss.adj), col, row,
      ~ flav[..] + gel[..] +
        flav.2[..] + gel.2[..] + flav.gel[..] +
        (1 | ..),
        data = springall)
```

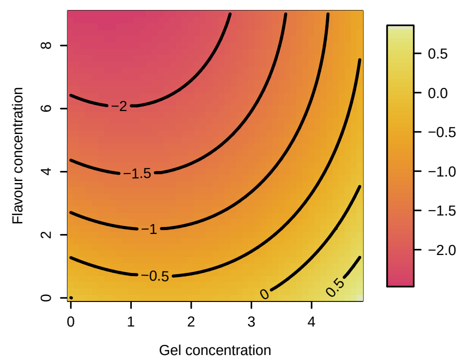
Extended Bradley-Terry models
└Implementation in R: The BradleyTerry2 package

Response Surface Model

```
Bradley Terry model fit by glmmPQL.fit
PQL algorithm converged to fixed effects model

Call:  BTm(outcome = cbind(win.adj, loss.adj), player1 = col,
player2 = row, formula = ~flav[..] + gel[..] + flav.2[..] +
gel.2[..] + flav.gel[..] + (1 | ..), data = springall)

Coefficients:
    flav[..]      gel[..]   flav.2[..]     gel.2[..]  flav.gel[..]
    -0.41194     -0.32578      0.01565       0.10506       0.02376

Degrees of Freedom: 36 Total (i.e. Null);  31 Residual
Null Deviance:      327.9
Residual Deviance: 15.47   AIC: 113
```

Extended Bradley-Terry models
└Implementation in R: The BradleyTerry2 package

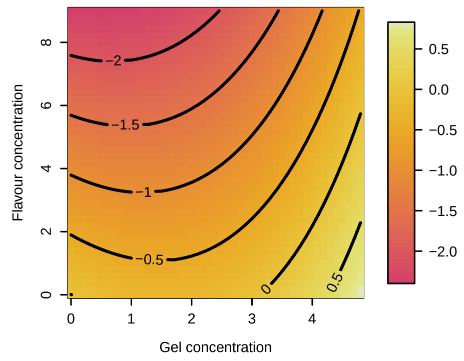
Second Order Response Surface



Simplified Model

```
> springall.model2 <- update(springall.model, ~ . - flav.2[..])
Bradley Terry model fit by glmmPQL.fit

Call:
BTm(outcome = cbind(win.adj, loss.adj), player1 = col, player2 = row,
    formula = ~flav[..] + gel[..] + gel.2[..] + flav.gel[..] +
        (1 | ..), data = springall)

Fixed effects:

    flav[..]       gel[..]     gel.2[..]  flav.gel[..]
    -0.26366      -0.32690       0.10416       0.02476

Random Effects Std. Dev.: 0.1406561
```


Fitted Response Surface



New work on ties (not yet in *BradleyTerry2*)

Davidson (1970) formulation:

$$\mathrm{pr(tie)} = \frac{\nu\sqrt{\alpha_i\alpha_j}}{\alpha_i + \alpha_j + \nu\sqrt{\alpha_i\alpha_j}}$$

$$\mathrm{pr}(i \text{ beats } j \,|\, \text{not tied}) = \frac{\alpha_i}{\alpha_i + \alpha_j}$$

For inference: either

- discard ties, use the conditional likelihood (robust?)
- ML for all parameters including ν (efficient?)

A log-linear model. But too restrictive?

$\nu \to \infty$: $\mathrm{pr(tie)} \to 1$

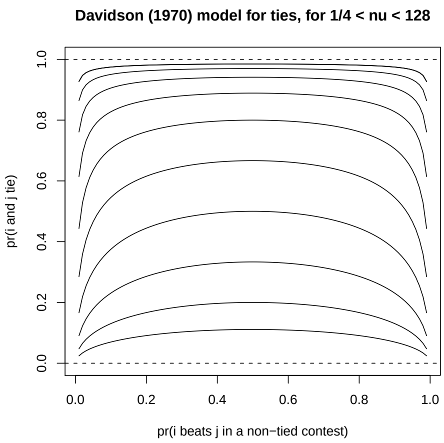
$\nu \to 0$: $\mathrm{pr(tie)} \propto \nu\sqrt{\alpha_i\alpha_j}/(\alpha_i + \alpha_j)$ (approx.)

The single extra parameter ν conflates

- overall (max) probability of a tie
- strength of dependence of $\mathrm{pr(tie)}$ on α_i, α_j.

And the strongest dependence allowed (i.e., as $\nu \to 0$) is actually rather weak.

(Same comments apply to the Rao-Kupper model for ties.)



A '2-parameter' model for ties

Details omitted here — paper in preparation, preprint to appear soon at `http://go.warwick.ac.uk/dfirth`