Statistical inference for the mixed Bradley-Terry model

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Ongoing joint work: D. Firth (Warwick) & C. Varin (Padova, Italy)

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Plan

Bradley-Terry models

Data structure

- N players
- Compare "player" i with "player" j in contest ijt, $\ t = 1, \ldots, T_{ij}$

Simplest version: Binary

$$y_{ijt} = \begin{cases} 1 & \text{if } i \text{ beats } j \\ 0 & \text{if } j \text{ beats } i \end{cases}$$

Elaborations: ties; margin of victory (ordinal or continuous)

Linear predictor: $\eta_{ijt} = g\left[Pr(y_{ijt} = 1)\right]$

- Pure B.T: $\eta_{ijt} = \alpha_i - \alpha_j$
- General: $\eta_{ijt} = \alpha_i - \alpha_j + z_{ijt}^T \gamma$

$$\text{e.g.} \qquad z_{ijt} = \begin{cases} 1 & i \text{ is at home} \\ -1 & j \text{ is at home} \\ 0 & \text{otherwise} \end{cases}$$

Some typical aims

1. *Rank* the players according to their ability score α_i
2. *Explain* ability in terms of player-specific covariates x_i:

$$\alpha_i = x_i^T \beta + u_i, \quad u_i \text{ iid } \sim F_u(\sigma)$$

Random effects

- *Aim 1:* $u_i = \alpha_i$

  ensures appropriate shrinkage of ability estimates to take account of *imprecision* of estimation (e.g., Efron and Morris, JASA, 1975)

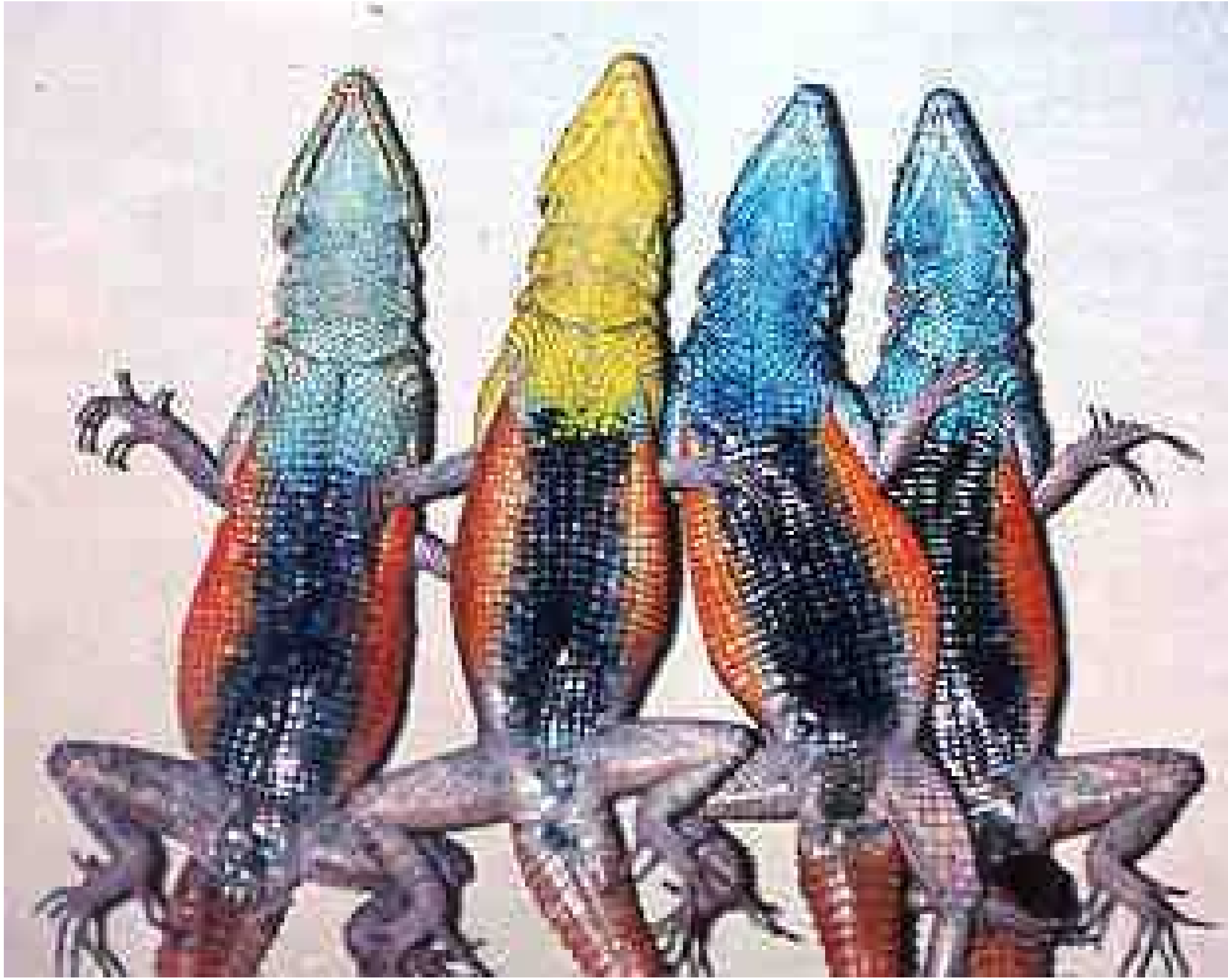
- *Aim 2:* $u_i = \alpha_i - x_i^T\beta$

  to represent *unexplained* variation in ability; often not recognised in the literature (e.g., Springall, 1973)

General model: The linear predictor has the form

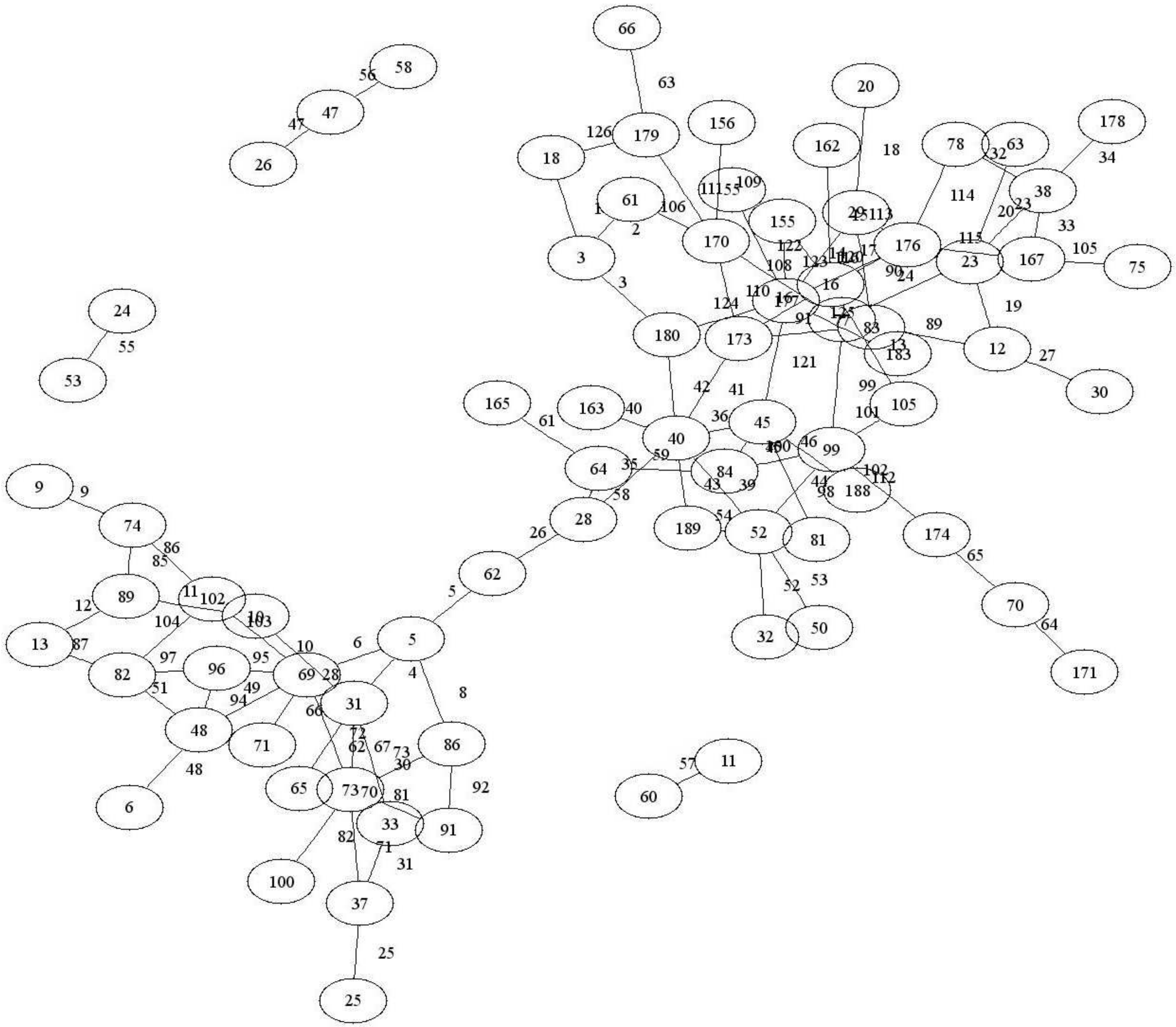
$$\eta_{ijt} = (x_i^T - x_j^T)\beta + u_i - u_j + z_{ijt}^T\gamma$$

Lizards data: *Whiting et al.* (2006)



- 189 male lizards are captured and explanatory variables are made
- Then released, and *contests* (fights) observed
- 100 contests (winner, loser) were observed involving 77 lizards

- Explanatory variables
  - ⋆ PC1throat, PC2throat, PC3throat: first 3 PCs of throat spectrum
  - ⋆ SVL: snout-vent length
  - ⋆ HL.res, HW.res, HH.res: residual of head length, width, height on SVL



Previous work [Whiting et al. (2006)] & [D. Firth (2005)]

$$g\left[Pr(y_{ij}=1)\right]=\sum_{r=1}^{4}(x_{ir}-x_{jr})\beta_r,$$

x_1=PC1throat, x_2 =PC3throat, x_3=SVL, x_4= HL.res

Present work

$$g\left[Pr(y_{ij}=1)\right]=\sum_{r=1}^{4}(x_{ir}-x_{jr})\beta_r+u_i-u_j,$$

$$u_i \sim N(0,\sigma^2)$$

Marginal likelihood

$$L(\beta,\sigma;y) \quad = \quad \int_{\mathbb{R}^{77}}\prod_{i=1}^{77}\prod_{j=i+1}^{77} Pr(y_{ij} \mid u_i,u_j)\varphi_{\sigma^2}(u)du$$

- *Belongs to the composite likelihood class* defined by Lindsay (1988)

Definition A rich class of pseudo likelihoods based on the composition likelihood type objects

Idea Choose a set of events, write the likelihood for each of them and then take the weighted product

Examples

1. Besag's pseudolikelihood (Besag, JRSSb, 1974)
2. Partial likelihood (Azzalini, BKA, 1983)
3. Composite marginal likelihood
    - Pairwise (Cox and Reid, 2004; Bellio and Varin, 2005)
    - Triplewise likelihood
    - Combination of both them

Motivation

1. Make likelihood type inference in complex models for dependent data
2. Reduce the computational effort and difficulties
3. Gain in statistical robustness with respect to full likelihood
4. In many applications, the cost in efficiency reduction relatively to the full likelihood is moderate
5. Under suitable regularity conditions, the MCLE is consistent and asymptotically normal

Based on all the observed pairs of contests with a common player: $(y_{ij}, y_{ij^\star})$

$$L_2(\beta, \sigma; y) = \prod_{i=1}^{N} \prod_{\{j<j^\star;\ j,j^\star \neq i\}} Pr\,(y_{ij}, y_{ij^\star}; \beta, \sigma)$$

$$Pr\,(y_{ij}, y_{ij^\star}; \beta, \sigma) \quad = \quad \int_{\mathbb{R}^3} Pr(y_{ij} \mid u_i, u_j) Pr(y_{ij^\star} \mid u_i, u_{j^\star}) \varphi(u_i, u_j, u_{j\star}) du_i du_j du_{j^\star}$$

1. **Probit link**:
   $Pr(y_{ij} = 1, y_{ij^\star} = 1) = \Phi_2 \left(\frac{(x_i - x_j)^T \beta}{\sqrt{1+2\sigma^2}}, \frac{(x_i - x_j^\star)^T \beta}{\sqrt{1+2\sigma^2}}; \frac{\sigma^2}{\sqrt{1+2\sigma^2}} \right)$
2. **logit link**: Using scale mixture of Drum and McCullagh (1993)

$$F(t) = \frac{e^t}{1+e^t} \simeq \sum_{i=1}^{k} p_{k,i} \Phi(t s_{k,i}),$$

where $(s_{k,i}, p_{k,i})$ are known for $k = 1, \ldots, 8$.

Simulation

- based on 300 data sets
- 2 sizes: $N = 20,\ 30$
- $\beta = (\beta_1, \beta_2) = (-1, 2)$
- $\sigma = 0.5,\ 1,\ 2$
- $x_1,\ x_2\ \sim\ N(0, 1)$

Interest: mean and standard deviation (sd) of estimators of β and σ

		N=20		N=30	
parameter	true	mean	sd	mean	sd
β_1	-1	-1.03	0.27	-1.03	0.17
β_2	2	2.09	0.39	2.06	0.26
σ	0.5	0.48	0.19	0.49	0.16
β_1	-1	-1.04	0.37	-1.02	0.29
β_2	2	2.10	0.55	2.08	0.41
σ	1	1.01	0.38	0.97	0.29
β_1	-1	-1.19	0.64	-1.12	0.39
β_2	2	2.31	0.95	2.10	0.57
σ	2	2.09	0.81	2.08	0.45

1. Structure
   - *Sparse data*: 77 lizards with only 100 contests in total
   - Two lizards with *missing values* in the covariates: liz096, liz099 (removed from the analysis)
   - Two lizards which *always win* (7 victories for each): liz040, liz073
2. Analysis with liz040 & liz073
   - *Problem* in the optimization due to *infinite values* (very frequent in binary data)
   - liz040, liz073 are the cause
3. Analysis without liz040 & liz073

   Probit link:

$$\hat{\beta} = (-0.12, 0.27, 0.25, -0.67), \quad \hat{\sigma} = 0.59, \quad l_2(\hat{\beta}, \hat{\sigma}) = -113.31$$

   Logit link:

$$\hat{\beta} = (-0.22, 0.49, 0.47, -1.27), \quad \hat{\sigma} = 1.14, \quad l_2(\hat{\beta}, \hat{\sigma}) = -113.20$$

Work in progress on:

1. Solve the problem of the infinite values
   - Optimality of the approach
     - ⋆ WPL: Weighted pairwise likelihood
     - ⋆ mixture of L_2 and L_1
   - Penalized pairwise likelihood
2. Bayesian version of the model using McMC

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